Lecture 5

Reaction-Diffusion Models as Dynamical Systems

Models are of the form (1.1), where L_i in (1.1) has the form (1.3), L_i strongly uniformly elliptic and meeting the hypotheses of Thm 4.3.

Such systems are called parabolic in the terminology of pde's. Important references in this discussion include:

Friedman (1964) (a priori estimates)
Friedman (1976)
Pazy (1983)
Henry (1981) (analytic semi-groups)
Amann (1988, 1989, 1990) (quasilinear problems)

Our sketch follows Mora (1983). (See also Cantrell, Cosner and Hutson (1993a).)

A local solution is one that exists on some time interval $[0, T)$

with $0 < T < \infty$. A <u>global</u> solution exists for all $t > 0$.

Generally speaking, the way that a local solution can fail to be global is by becoming infinite in finite time.

<u>Sketch of the ideas behind results</u>

(i) We have a pair of Banach spaces X, Y with

$X \subseteq Y$ and a linear operator $A : X \to Y$

with certain properties including the continuity of

A and $(A - \lambda I)^{-1}$ for $\lambda \in \mathbb{C}$

with $\operatorname{Re} \lambda \leq \Lambda_0$

(Here typically

$$Y = \left[L^p(\Omega)\right]^m$$

$$X = \{u \in [W^{2,p}(\Omega)]^m \mid B_i u_i = 0 \text{ on } \partial\Omega, \ i = 1, \dots, m\})$$

(components of elements of X, Y are complex-valued functions)

(ii) If need be, we add a multiple of u_i (say $c_i u_i$) to $f_i(x, u_1, \dots, u_m)$ and subtract from L_i so that each L_i in (1.1) satisfies hypotheses of Thm 4.5/4.6.

(iii) Then A will be the $m \times m$ diagonal matrix of operators with $-L_i$ in the i,i position, $i = 1, \ldots, m$.

(iv) Thm 4.5 / Thm 4.6 $\Rightarrow$ A^{-1} exists as a bounded continuous linear operator from Y into X.

A priori estimates (e.g.

$$\|u\|_{2,p} \leq C\left(\|u\|_{0,p} + \|L_i u\|_{0,p}\right)$$

(5.1) $$\Rightarrow \quad \|(A+\lambda I)^{-1} u\|_Y \leq C \|u\|_Y / |\lambda|$$

for $\lambda \in \mathbb{C}$ with $|\lambda| \geq R_0$ for some fixed R_0

and $-\pi/2 - \delta < \arg\lambda < \pi/2 + \delta$

for some $\delta > 0$ (Friedman 1976)

(v) From (5.1) we obtain that $-A$ generates an analytic semigroup on Y (usually denoted e^{-tA}), where by a semi-group of linear operators we mean a family $S(t)$ of operators defined for $t \geq 0$

and depending continuously on t so that for each $t \geq 0$, $S(t)$ is a bounded linear operator from Y to Y, with

$$S(0) = I$$

$$S(s)S(t) = S(s+t)$$

(The operator $-A$ is called the infinitessimal generator of e^{-tA}. The defining property of the infintessimal generator B of a semi-group $S(t)$ is

$$\lim_{h \to 0} \frac{[S(h)u - u]}{h} = Bu$$

for all $u \in \operatorname{dom} B$, where the limit is taken in the metric induced by the norm of Y.)

(Again recall Friedman (1976), Pazy (1983), Henry (1981).)

The key properties of e^{-tA} are that it defines a semi-group of bounded linear operators on Y, which depend analytically on t for $t \in \mathbb{C}$ with $|\arg t| < \delta$,

and for any such t, the operators

$$d(e^{-tA})/dt \, , \quad A e^{-tA}$$

are also continuous on Y, with

$$\frac{d(e^{-tA} u)}{dt} = A e^{-tA} u$$

for any $u \in Y$. Further, for $t \in \mathbb{C}$ with $|\arg t| < \delta$, we have

(5.2) $$\| A e^{-tA} u \|_Y \le (C/|t|) \|u\|_Y$$

for $u \in Y$, C independent of u.

(Friedman (1976), Pazy (1983), Henry (1981))
(Schechter as well)

(vi) We write (1.1) in the abstract form

(5.3) $$\frac{du}{dt} = -Au + F(u)$$

$$u(0) = u_0 \in Y$$

where $F(u)$ is the matrix with terms $f_i(x, u_1, \ldots, u_m)$

on the diagonal and zeros elsewhere. We express the solution as

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} F(u(s))\,ds \tag{5.4}$$

An issue is to make clear that (5.4) makes sense in the context of (5.3). If we apply A to both sides of (5.4) we get

$$Au = Ae^{-tA} u_0 + \int_0^t A e^{-(t-s)A} F(u(s))\,ds$$

All we get from (5.2) about the integrand is

$$\| A e^{-(t-s)A} F(u(s)) \| \leq \frac{c}{|t-s|} \|F(u(s))\|_Y$$

But $\int_0^t \frac{1}{|t-s|}$ is not integrable!

(vii) Operators which generate analytic semi-groups can be analysed using techniques from complex variable theory and the theory of Laplace transforms, which enable us to define so-called <u>fractional</u>

<u>powers</u> A^γ of A, for which

$$A^{-\alpha} u = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha - 1} e^{-As} u \, ds$$

$$A^\alpha = (A^{-\alpha})^{-1}$$

$$D(A^\alpha) = R(A^{-\alpha})$$

(5.2) is replaced by

$$\|A^{\gamma} e^{-tA} u\|_Y \leq (C/|t|^{\gamma}) \|u\|_Y$$

> So if we apply A^{γ} (instead of just A) to (5.3) we get

(5.5) $$A^{\gamma} u = A^{\gamma} e^{-tA} u_0 + \int_0^t A^{\gamma} e^{-(t-s)A} F(u(s))\, ds$$

For $0 < \gamma < 1$ the integral term is well-behaved if $F(u)$ is bounded in Y, since $\int_0^t \left[\frac{1}{(t-s)^{\gamma}}\right] ds < \infty$ for $0 < \gamma < 1$.

(viii) Furthermore, we can define Banach spaces $X^{\gamma} \subseteq Y$ by taking X^{γ} to be the completion of X in Y with respect to the norm

$$\|u\|_{\gamma} = \|A^{\gamma} u\|_Y$$

Since A^{-1} is a compact operator on Y, it follows that the embedding of X^{γ_1} into X^{γ_2} is compact if $\gamma_1 > \gamma_2$.

(ix) At this point we can use (5.3) to define a semi-flow on X^γ for $\gamma \in (0,1)$ provided $F(u(s))$ is smooth in much the same way as we would for systems of ODE's. (F, P, H)

(x) If we know that orbits which are bounded in X^{γ_2} are also bounded in X^{γ_1} for some $\gamma_1 > \gamma_2$, we may conclude that orbits in X^{γ_2} are pre-compact (since $X^{\gamma_1} \hookrightarrow X^{\gamma_2}$ compact)

What we actually get are bounds

$$\|e^{-tA} u\| \le C e^{-\beta t} \|u\|_Y$$

$$\|A^\gamma e^{-tA} u\| \le C \left(e^{-\beta t} / |t|^\gamma \right) \|u\|_Y$$

which gives bounds on $\|u\|_\gamma$ when $F(u)$ is bounded in Y.
(F, P, H)

(xi) In practice we usually aim for bounds on $\vec{u}$ in $[C(\bar{\Omega})]^m$. Such bounds imply bounds for $f_i(x, u_1, \ldots, u_m)$ in $L^p(\Omega)$ for any $p \in [1, \infty]$

if the f_i are smooth. So we can translate bounds on $\vec{u}$ in $[C(\bar{\Omega})]^m$ to conclude that orbits for the system are bounded in X^γ for any $\gamma \in (0,1)$ and hence pre-compact in X^γ.

(F, P, H)

(xii) (a) What does $\vec{u} \in X^\gamma$ mean in terms of the smoothness of $\vec{u}$?

(b) Are solutions to (5.3) classical solutions to (1.1)?

For (a), if we start with $Y = [L^p(\Omega)]^m$ and $X \subseteq [W^{2,p}(\Omega)]^m$, X^γ embeds into $[C^{1+\alpha}(\bar{\Omega})]^m$ provided $0 < \alpha < 2\gamma - (n/p) - 1$

(Pazy 1983)

For (b), we need to assume $f_i(x,u)$ are Hölder continuous in x and at least Lipschitz in u. In that case a priori estimates for parabolic equations analogous to Thms 4.1, 4.2, 4.5, 4.6 for elliptic equations guarantee the solutions are classical. (Included in notes for completeness.)

(Friedman 1964)

Formal statements of Theorems

Theorem 5.1 Suppose that the domain Ω, operators L_i and boundary conditions in (1.1) satisfy the hypotheses of Theorem 4.3 for $i=1,\dots,m$. Suppose for each $i \in \{1,\dots,m\}$, the function $f_i(x,u_1,\dots,u_m) = f_i(x,\vec{u})$ is measurable and bounded uniformly in x in $\bar{\Omega}$ when $\vec{u}$ is restricted to a bounded subset of $\mathbb{R}^m$, and that $f_i(x,\vec{u})$ is Lipschitz continuous in $\vec{u}$, uniformly for $x \in \bar{\Omega}$ and $\vec{u}$ restricted to any bounded subset of $\mathbb{R}^m$.

Let $Y = [L^p(\Omega)]^m$ for some $p > n$ and let $X = \{\vec{u} \in [W^{2,p}(\Omega)]^m \mid B_i u_i = 0$ on $\partial\Omega$, $i=1,\dots,m\}$.

Let A be the matrix of operators $-L_i$ as the ith diagonal element and with all off-diagonal elements equal to 0. Let $X^\gamma \subseteq Y$ denote the space generated by A^γ for $\gamma \in (0,1)$, so that

$$\|u\|_{X^\gamma} = \|A^\gamma u\|_Y$$

Then there exists a $\gamma_0 \in (0,1)$ such that for $\gamma \in (\gamma_0, 1)$, the system (1.1) generates a local semi-flow on X^γ. Bounded orbits in X^γ are pre-compact.

Notes: Henry ((1981)

Key points in proof:

(i) A priori estimates of Agmon et al (1959) $\Rightarrow$

$-A$ generates an analytic semi-group on $[W^{2,p}(\Omega)]^m$

(ii) General theory of analytic semi-groups allows us to re-cast (1.1) as (5.3) and then to define X^γ

(iii) Since $p > n$, we can choose $\gamma_0 \ni$

$\gamma \in (\gamma_0, 1) \Rightarrow 0 < 2\gamma - (n/p) - 1$

$\Rightarrow X^\gamma \hookrightarrow [C^{1+\alpha}(\bar\Omega)]^m$ (Pazy 1983)

(iv) Hypotheses on $f_i \Rightarrow$ if $\vec{u} \in [C^{1+\alpha}(\bar\Omega)]^m$,

$f_i \in L^p(\Omega)$ for $p \geq 1 \Rightarrow (f_1, \ldots, f_m) \in Y$

$\Rightarrow$ (5.5) is defined and (5.3) defines a

semi-dynamical system on X^γ

(v) Since $(\lambda I + A)^{-1}$ exists as a linear compact

operator on $[L^p(\Omega)]^m$ for λ real and

sufficiently large, the embeddings $X^{\gamma_1} \hookrightarrow X^{\gamma_2}$

for $\gamma_1 > \gamma_2 > 0$ are compact, and hence

we get boundedness of compact orbits.

We may prefer to work in the more concrete spaces

$C(\bar\Omega)$ or $C^1(\bar\Omega)$. What will we need to do so?

<u>Theorem 5.2</u> (Mora, 1983). Suppose the hypotheses

of Theorem 5.1 hold. Suppose in addition that

f_i is C^2 in x and $\vec{u}$; i.e., $f_i \in C^2(\bar\Omega \times \mathbb{R}^m)$

for each $i \in \{1, \ldots, m\}$.

Then (1.1) generates a local semi-flow on $[C(\bar{\Omega})]^m$ under Neumann boundary conditions. Under Dirichlet boundary conditions, (1.1) generates a local semi-flow on the subspaces of $[C(\bar{\Omega})]^m$ and of $[C^1(\bar{\Omega})]^m$ consisting of functions which are zero on $\partial\Omega$. Under Neumann or Robin boundary conditions, (1.1) generates a local semi-flow on the subspace of $[C^1(\bar{\Omega})]^m$ consisting of functions satisfying the boundary conditions. In each case, bounded orbits are pre-compact.

Global solutions

So far our results on reaction-diffusion models as semi-dynamical systems are local in time. Some additional information is needed to conclude that a given orbit exists globally in time. For a semi-dynamical system on a Banach space Z in which bounded orbits are pre-compact it suffices to show that for each $T > 0$ there is a $B(T) < \infty$ so that

$$\sup \{ \|\vec{u}(t)\|_Z : 0 \leq t < T \} \leq B(T)$$

to conclude that the orbit $u(t)$ exists for all $t > 0$.

In the models we consider it will suffice to show that

$$\sup \{ |\vec{u}(x,t)| : x \in \bar{\Omega},\ 0 \leq t < T \} \leq B(T)$$

because the bound on $|\vec{u}| \Rightarrow$ a bound in $[L^p(\Omega)]^m$ for any $p \in [1,\infty)$. From there, (5.5) or analogues $\Rightarrow$ boundedness in X^γ, which play the role of state spaces for the semi-dynamical system or which embed in $[C^{1+\alpha}(\bar{\Omega})]^m$, which in turn embeds in state spaces constructed from $[C(\bar{\Omega})]^m$ or $[C^1(\bar{\Omega})]^m$. In many cases, it is straightforward to get bounds on $|\vec{u}|$. However, there are some cases, such as predator-prey systems where we do not impose a quadratic mortality penalty on the predator, where we have to work fairly hard to get the required estimates.

Classical Regularity for Parabolic Equations

For functions on $\bar{\Omega} \times [0,T]$, define

$$[f]_{\alpha, \alpha/2} = \sup_{\substack{(x,t),(y,s) \in \bar{\Omega} \times [0,T] \\ (x,t) \neq (y,s)}} \left(\frac{|f(x,t) - f(y,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}} \right)$$

$$C^{\alpha, \alpha/2}(\bar{\Omega} \times [0,T]) = \left\{ f \in C(\bar{\Omega} \times [0,T]) \,\middle|\, [f]_{\alpha, \alpha/2} < \infty \right\}$$

$C^{\alpha, \alpha/2}(\bar{\Omega} \times [0,T])$ is a Banach space with norm

$$\|u\|_{\alpha, \alpha/2} = \sup_{\bar{\Omega} \times [0,T]} |u(x,t)| + [u]_{\alpha, \alpha/2}$$

Let $\beta = (\beta_1, \dots, \beta_n)$ be the multi-index and define ∂_x^β as before. Let ∂_t denote the derivative wrt t.

$C^{2+\alpha,\,1+\alpha/2}(\bar{\Omega}\times[0,T])$ = the space of functions on $\bar{\Omega}\times[0,T]$,

whose derivatives up to order two x and order 1 in t are

Hölder continuous, with norm

$$\|u\|_{2+\alpha,\,1+\alpha/2}$$

$$= \sup_{(x,t)\in\bar{\Omega}\times[0,T]} |u(x,t)| + \sum_{|\beta|\le 2} \sup_{(x,t)\in\bar{\Omega}\times[0,T]} |\partial_x^\beta u(x,t)|$$

$$+ \sup_{(x,t)\in\bar{\Omega}\times[0,T]} |\partial_t u(x,t)| + \sum_{|\beta|=2} [\partial_x^\beta u(x,t)]_{\alpha,\alpha/2} + [\partial_t u(x,t)]_{\alpha,\alpha/2}$$

These spaces are Banach (Friedman 1964)

We also require the space $C^{1+\delta,\,\delta/2}(\bar{\Omega}\times[0,T])$ whose

norm is given by

$$\|u\|_{1+\delta,\,\delta/2} = \|u\|_{\delta,\,\delta/2} + \sum_{|\beta|=1} \|\partial_x^\beta u\|_{\delta,\,\delta/2}$$

Theorem 5.3 (Friedman 1964). Suppose that Ω is a bounded domain

with $\partial\Omega$ of class $C^{2+\alpha}$ and that L is an elliptic operator

of form (1.3), which is strongly uniformly elliptic, here allowing coefficients a_{ij}, b_i and c to depend on t as well as x.

Suppose $a_{ij}, b_i, c \in C^{\alpha, \alpha/2}(\bar{\Omega} \times [0,T])$. If $f(x,t) \in C^{\alpha, \alpha/2}(\bar{\Omega} \times [0,T])$, $g(x,t) \in C^{2+\alpha, 1+\alpha}(\bar{\Omega} \times [0,T])$ and $Lg = f$ for $t = 0$, $x \in \partial\Omega$, then the problem

$$(5.6) \qquad \frac{\partial u}{\partial t} - Lu = f(x,t) \qquad \text{in } \Omega \times (0,T]$$

$$u(x,t) = g(x,t) \qquad \text{on } \partial\Omega \times [0,T] \cup (\Omega \times \{0\})$$

has a unique solution $u \in C^{2+\alpha, 1+\alpha/2}(\bar{\Omega} \times [0,T])$ with

$$\|u\|_{2+\alpha, 1+\alpha/2} \le C\left(\|f\|_{\alpha, \alpha/2} + \|g\|_{2+\alpha, 1+\frac{\alpha}{2}}\right)$$

where C is independent of f and g.

Theorem 5.4 (Ladyzhenskaya et al 1968). Suppose that Ω, L and f satisfy the hypotheses of Theorem 5.3. Suppose that $g(x) \in C^{2+\alpha}(\bar{\Omega})$ and $\gamma(x)$, $\beta(x)$ and $h(x)$

$\in C^{1+\alpha}(\partial\Omega)$, with $\gamma \geq 0$ and $\beta > 0$ on $\partial\Omega$.

Suppose also that

$$\gamma(x)\, g(x) + \beta(x) \frac{\partial g}{\partial \eta} = h(x)$$

on $\partial\Omega$.

Then the problem

$$\frac{\partial u}{\partial t} - Lu = f(x,t) \qquad \text{in } \Omega \times (0,T]$$

$$u(x,0) = g(x) \qquad \text{in } \bar{\Omega}$$

$$\gamma(x)\, u(x,t) + \beta(x) \frac{\partial u}{\partial \eta}(x,t) = h(x) \qquad \text{on } \partial\Omega \times [0,T]$$

has a unique solution in $C^{2+\alpha,\, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [0,T])$

so that

$$\|u\|_{2+\alpha,\, 1+\frac{\alpha}{2}} \leq C\left(\|f\|_{\alpha,\frac{\alpha}{2}} + \|g\|_{2+\alpha} + \|h\|_{1+\alpha}\right)$$

where the norm h is taken in $C^{1+\alpha}(\partial\Omega)$, the norm of g is taken in $C^{2+\alpha}(\bar{\Omega})$, and C is independent of f, g, h.

Lemma 5.5 (Friedman 1964). Suppose Ω and L satisfy the hypotheses of Theorem 5.3. Suppose in addition that the coefficients a_{ij} are uniformly Lipschitz in $\bar{\Omega} \times [0,T]$. Suppose that $f(x,t)$ is continuous on $\bar{\Omega} \times [0,T]$ with $f(x,0) = 0$ on $\partial\Omega$.

Then for any $\delta \in (0,1)$, there is a constant C independent of $f(x,t)$ so that any solution u of

$$\frac{\partial u}{\partial t} - Lu = f(x,t) \quad \text{in } \Omega \times (0,T]$$

$$u(x,t) = 0 \quad \text{on } (\partial\Omega \times (0,T]) \cup (\bar{\Omega} \times \{0\})$$

satisfies $$\|u\|_{1+\delta, \delta/2} \le C \sup_{(x,t) \in \bar{\Omega} \times [0,T]} |f(x,t)|$$

An analogue holds if the Dirichlet condition is replaced by

$$\gamma(x) u(x) + \beta(x) \frac{\partial u}{\partial \eta}(x) = 0$$

on $\partial\Omega \times (0,T]$ with γ and β as in Theorem 5.4

Theorem 5.3 / Lemma 5.5 and Theorem 5.4 / (Analogue to Lemma 5.5)

$\Rightarrow$ trajectories of semi-dynamical systems whose existence is asserted in Theorem 5.1 correspond to classical solutions of (1.1) if Ω, f_i, and the coefficients of L_i and B_i are sufficiently smooth. Note that the f in (5.6) corresponds to

$$f_i(x, u_1(x,t), \ldots, u_m(x,t))$$

and that the embedding properties of X^γ enable us to satisfy the hypotheses on f in Theorem 5.3. We can apply Theorem 5.3 equation by equation in (1.1) because we don't have coupling in derivative terms (i.e. we are discussing <u>semi-linear</u> versus quasi-linear problems here.)